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REGULARIZED NUMERICAL SOLUTION OF THE NONLINEAR, TWO-DIMENSIONAL, INVERSE HEAT-CONDUCTION PROBLEM

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Inverse heat-conduction (IHC) methods are used extensively to solve problems in the diagnosis and identification of heat-transfer processes from experimental results [1-4]. However, methods for the solution of one-dimensional IHC problems may have been studied more or less completely so far, but methods for the solution of nonlinear two-dimensional and three-dimensional IHC methods are in need of further refinement. The current literature is devoted mainly to the solution of linear two-dimensional IHC boundary-value problems. The best-known results in this area have been obtained by O. M. Alifanov et al., who have developed an iterative regularization method. Two- and three-dimensional IHC problems are formulated in [1], where a method is also proposed for the iterative solution of the linear two-dimensional inverse problem in an extremal setting for bodies in the form of flat plates, admitting generalization to other geometries. Alifanov and Kerov [5, 6] discuss a procedure and algorithm for the iterative solution of the linear two-dimensional IHC problem with finite-difference approximation of the heat-conduction boundary-value problem in the case of a hollow circular cylinder and a cylindrical copper shell, and they give the results of a methodological study of their algorithm. This approach has been elaborated [7] in application to the integral form of the two-dimensional problem with constant thermophysical coefficients. A major breakthrough in regard to the iterative regularization method is reported in [8], where the method is used to construct an algorithm for solving the three-dimensional inverse boundary-value problem for a multilayered, hollow, spherical segment. Unfortunately, the proposed algorithm has not been checked out numerically in [8].

In implementing the iterative regularization method, the greatest difficulties are encountered in calculating the gradient of the objective functional. This procedure requires considerable ingenuity on the part of the investigator and can pose an almost intractable problem for complex mathematical models. Numerical regularizing methods could be useful for solving the inverse problem in the given situation. A Tikhonov-regularizing algorithm has been proposed [9] for the numerical solution of the nonlinear, one-dimensional IHC problem. This algorithm has subsequently been elaborated and used [10-13] to solve specific problems in the mechanics of reacting media.

In this article we generalize the one-dimensional regularizing algorithm to the two-dimensional case. We investigate the influence of heat flow on the accuracy of determination of temperature and heat flux density by IHC methods. We also demonstrate the influence of initial data error on the solution of the inverse problem.

1. Physical and Mathematical Statements of the Inverse Problem. We formulate the IHC problem for a body of rectangular cross section (Fig. 1). We assume that the heat flux vector is parallel to the xy plane at every point in space. Heat transfer takes place in the planes $x = 0$, $x = b$, and $y = d$, being specified by boundary conditions of the Dirichlet, Neumann, or Cauchy type. It is required to determine the heat flux density $q_w(x, y)$ and the temperature $T_w(x, t)$ at the boundary (wall) $y = 0$ from the known temperature at the line $y = c$.

This IHC problem is stated mathematically in the form

$$C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x(T)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y(T)\frac{\partial T}{\partial y}\right),\quad 0 < x < b,\ 0 < y < d,\ t_b < t \leq t_f; \tag{1.1}$$

$$T(x, y, t_b) = T_b(x, y),\ 0 \leq x \leq b,\ 0 \leq y \leq d; \tag{1.2}$$

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$$r_1(y, t)\frac{\partial T(0, y, t)}{\partial x} + r_2(y, t)T(0, y, t) = r_3(y, t); \quad (1.3)$$

$$s_1(y, t)\frac{\partial T(b, y, t)}{\partial x} + s_2(y, t)T(b, y, t) = s_3(y, t); \quad (1.4)$$

$$g_1(x, t)\frac{\partial T(x, d, t)}{\partial y} + g_2(x, t)T(x, d, t) = g_3(x, t); \quad (1.5)$$

$$T(x, c, t) = T_c(x, t),\ 0 \leqslant x \leqslant b,\ 0 \leqslant y \leqslant d,\ t_b \leqslant t \leqslant t_f; \quad (1.6)$$

$$q_w(x, t) = -\lambda_y(T(x, 0, t))\frac{\partial T(x, 0, t)}{\partial y} - ? \quad (1.7)$$

Here T is the temperature, x and y are the space coordinates, t is the time, C is the volumetric heat capacity, λ_x and λ_y are the thermal conductivities in the x and y directions, and r_1, r_2, r_3, s_1, s_2, s_3, g_1, g_2, and g_3 are coefficients characterizing the type of boundary conditions at the boundaries of the rectangular domain. For example, if $r_1 = s_1 = g_1 = 1$ and $r_2 = r_3 = s_2 = s_3 = g_2 = g_3 = 0$, adiabatic conditions prevail at the boundaries $x = 0$, $x = b$, and $y = d$. Subscripts: b) initial state; f) final state; w) heated boundary $y = 0$; c) interior line $y = c$.

2. Algorithm for Solving the Inverse Problem. The solution of the inverse problem is divided into two stages. In the first stage it is reduced to the Cauchy problem. The initial condition (1.2) and the boundary conditions (1.3)-(1.6) in the domain $D_2\{0 \le x \le b, c \le y \le d, t_b \le t \le t_f\}$ are used to find the temperature field and, as a result, the heat flux density $q_c(x, t) = -\lambda_y T(x, c, t)\partial T(x, c, t)/\partial y$ on the line $y = c$. This is a well-studied two-dimensional boundary-value problem, which can be solved by any standard numerical method, for example, the decoupling method [14]. The one-dimensional heat-conduction equations obtained by decoupling in each time half-step are efficiently solved by an iterative interpolation method [4]. In the second stage the inverse problem of determining the temperature field and the functions $T_w(x, t) = T(x, 0, t)$ and $q_w(x, t) = -\lambda_y T(x, 0, t)\partial T(x, 0, t)/\partial y$ from the known initial condition (1.2), the boundary conditions (1.3) and (1.4), and the functions $T_c(x, t)$ and $q_c(x, t)$ is solved in the domain $D_1(0 \le x \le b, 0 \le y \le c, t_b \le t \le t_f)$:

$$C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x(T)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y(T)\frac{\partial T}{\partial y}\right), \quad 0 < x < b,\ 0 < y < c,\ t_b < t \leqslant t_f; \quad (2.1)$$

$$T(x, y, 0) = T_b(x, y),\ 0 \leqslant x \leqslant b,\ 0 \leqslant y \leqslant c; \quad (2.2)$$

$$r_1(y, t)\frac{\partial T(0, y, t)}{\partial x} + r_2(y, t)T(0, y, t) = r_3(y, t); \quad (2.3)$$

$$s_1(y, t)\frac{\partial T(b, y, t)}{\partial x} + s_2(y, t)T(b, y, t) = s_3(y, t); \quad (2.4)$$

$$-\lambda_y(T(x, c, t))\frac{\partial T(x, c, t)}{\partial y} = q_c(x, t); \quad (2.5)$$

$$T(x, c, t) = T_c(x, t),\ 0 \leqslant x \leqslant b,\ 0 \leqslant y \leqslant c,\ t_b \leqslant t \leqslant t_b. \quad (2.6)$$

We introduce the differencing grid

$$h_x,\ h_y,\ h_t(x_l = h_x l,\ l = \overline{0, L};\ y_f = h_y k,\ k = \overline{0, K};\ t_j = h_t j,\ j = \overline{0, M}), \quad (2.7)$$

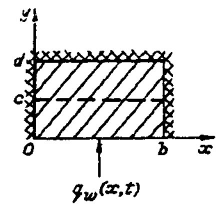

Fig. 1

where h_x, h_y, and h_z are the steps (increments) of the variables x, y, and t, respectively. To obtain the differencing scheme, we approximate the derivatives in the form

$$
\begin{gathered}
\left(\frac{\partial T}{\partial t}\right)^j_{l,k+1} \approx \frac{T^j_{l,k+1} - T^{j-1}_{l,k+1}}{h_t},\quad \left(\frac{\partial T}{\partial y}\right)^j_{l,k+1} \approx \frac{T^j_{l,k+2} - T^j_{l,k}}{2h_y},\\
\left(\frac{\partial T}{\partial x}\right)^j_{l,k+1} \approx \frac{T^j_{l+1,k+1} - T^j_{l-1,k+1}}{2h_x},\quad \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right)^j_{l,k+1} \approx\\
\approx \frac{1}{2h_y^2}\left[(\lambda^j_{yl,k+2} + \lambda^j_{yl,k+1})(T^j_{l,k+2} - T^j_{l,k+1}) - (\lambda^j_{yl,k+1} + \lambda^j_{yl,k})\right.\\
\left.\times (T^j_{l,k+1} - T^j_{l,k})\right],\quad \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right)^j_{l,k+1} \approx \frac{1}{2h_x^2}\left[(\lambda^j_{xl+1,k+1} + \lambda^j_{xl,k+1})\right.\\
\left.\times (T^j_{l+1,k+1} - T^j_{l,k+1}) - (\lambda^j_{xl,k+1} + \lambda^j_{xl-1,k+1})(T^j_{l,k+1} - T^j_{l-1,k+1})\right].
\end{gathered}
\tag{2.8}
$$

As a result, we obtain a nonlinear recursive relation for determining the temperature at the (l, k)-th spatial node:

$$
B^j_{l,k} T^j_{l,k} = F^j_{l,k},\quad l = \overline{1, L-1},\ j = \overline{1, M}.
\tag{2.9}
$$

Here

$$
\begin{gathered}
B^j_{l,k} = \frac{\lambda^j_{yl,k+1} + \lambda^j_{yl,k}}{2h_y^2};\quad F^j_{l,k} = C^j_{l,k+1} \frac{T^j_{l,k+1} - T^{j-1}_{l,k+1}}{h_t}\\
- \frac{1}{2h_x^2}\left[(\lambda^j_{xl+1,k+1} + \lambda^j_{xl,k+1})(T^j_{l+1,k+1} - T^j_{l,k+1}) - (\lambda^j_{xl,k+1} + \lambda^j_{xl-1,k+1})\right.\\
\left.\times (T^j_{l,k+1} - T^j_{l-1,k+1})\right] - \frac{1}{2h_y^2}\left[(\lambda^j_{yl,k+2} + \lambda^j_{yl,k+1})(T^j_{l,k+2} - T^j_{l,k+1})\right.\\
\left.- (\lambda^j_{yl,k+1} + \lambda^j_{yl,k}) T^j_{l,k+1}\right].
\end{gathered}
$$

The nonlinear equation (2.9) interrelates the k-th, (k + 1)-st, and (k + 2)-nd spatial lines. To begin the computational process, it is necessary to know the temperature on the K-th and (K − 1)-st lines. The temperature on the K-th line is given by the experimental function $T_c(x, t)$ from condition (2.6), and the temperature on the (K − 1)-st line is determined from the finite-difference analog of Eq. (2.5). Unless regularizers of some kind are introduced, Eq. (2.9) implements one of several direct numerical methods for the solution of IHC problems. In this case the temperature obtained in the k-th step is improved by iterations on the coefficients. Equation (2.9) can be used to obtain a regular solution of the inverse problem when the input temperature $T_c(x, t)$ has small fluctuation errors and the integration step with respect to the time h_t is sufficiently large. If the errors of the input temperature and the time step do not meet these restrictions, problem (2.9) does not give a stable solution. The capabilities of direct numerical methods can be extended by smoothing the input temperature.

Thus, when the direct numerical method is stable, a solution of the IHC problem cannot be obtained, and regularizing methods must be used.

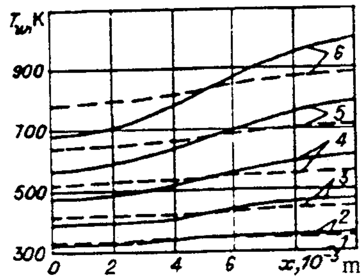

Fig. 2

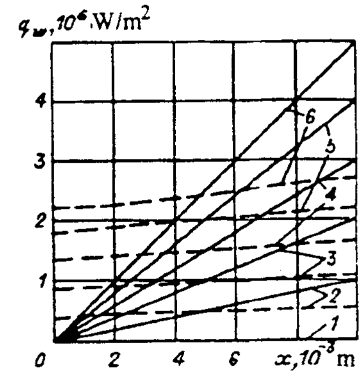

Fig. 3

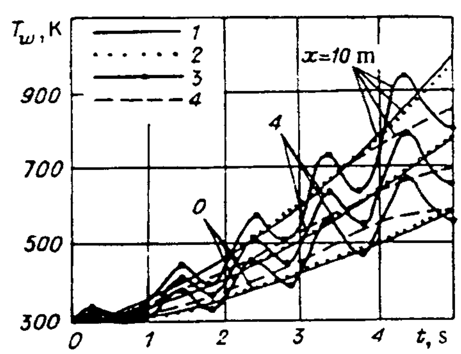

Fig. 4

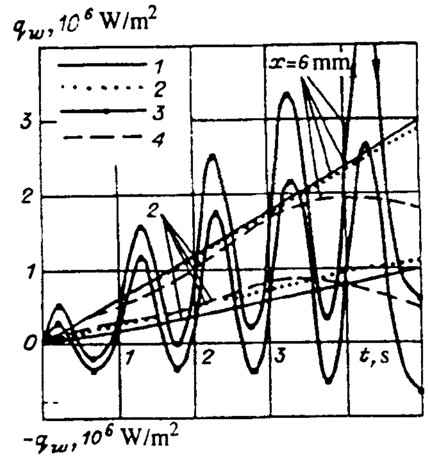

Fig. 5

For Eq. (2.9) we write the Tikhonov functional in the form

$$\Phi_{l,k}(\alpha)=\sum_{j=1}^{M}(B^j_{l,k}T^j_{l,k}-F^j_{l,k})^2+\frac{\alpha k_1}{h_t^2}\sum_{j=1}^{M}(T^j_{l,k}-T^{j-1}_{l,k})^2+$$
$$\frac{\alpha k_2}{h_t^4}\sum_{j=1}^{M-1}(T^{j+1}_{l,k}-2T^j_{l,k}+T^{j-1}_{l,k})^2+\alpha k_2 C_M^2,\ l=\overline{1,L-1}, \tag{2.10}$$

where α is a regularization parameter, $k_1 > 0$ and $k_2 > 0$ are nonnegative numbers, and $C_M = \partial^2 T_{l,k}(t_k)/\partial t^2$.

Minimizing Eq. (2.10) over all $T^j_{l,k}$($j = \overline{1,\ldots,M}$), we obtain a system of nonlinear algebraic equations with a symmetric, five-diagonal, positive definite matrix for finding a regularized solution at the (l, k)-th spatial node:

$$\sum_{\substack{i=j-2\\(1\leq i\leq M)}}^{j+2} a_{j,i}T^i_{l,k}=b_j,\ j=\overline{1,M},\ l=\overline{1,L-1}. \tag{2.11}$$

Here

$$a_{j,j} = \begin{cases} (B^1_{l,k})^2 + \alpha\left(2\dfrac{k_1}{h_t^2} + 5\dfrac{k_2}{h_t^4}\right), \ j = 1, \\ (B^j_{l,k})^2 + \alpha\left(2\dfrac{k_1}{h_t^2} + 6\dfrac{k_2}{h_t^4}\right), \ j = \overline{2, M-2}, \\ (B^{M-1}_{l,k})^2 + \alpha\left(2\dfrac{k_1}{h_t^2} + 5\dfrac{k_2}{h_t^4}\right), \ j = M-1, \\ (B^M_{l,k})^2 + \alpha\left(\dfrac{k_1}{h_t^2} + \dfrac{k_2}{h_t^4}\right), \ j = M; \end{cases}$$

$$a_{j,j+1} = a_{j+1,j} = \begin{cases} -\alpha\left(\dfrac{k_1}{h_t^2} + 4\dfrac{k_2}{h_t^4}\right), \ j = \overline{1, M-2}, \\ -\alpha\left(\dfrac{k_1}{h_t^2} + 2\dfrac{k_2}{h_t^4}\right), \ j = M-1; \end{cases}$$

$$a_{j,j+2} = a_{j+2,j} = \alpha\frac{k_2}{h_t^4}, \ j = \overline{1, M-2};$$

$$b_1 = B^1_{l,k}F^1_{l,k} + T_{bl,k}\alpha\left(\frac{k_1}{h_t^2} + 2\frac{k_2}{h_t^4}\right);$$

$$b_2 = B^2_{l,k}F^2_{l,k} - T_{bl',k}\alpha\frac{k_2}{h_t^4};$$

$$b_j = B^j_{l,k}F^j_{l,k}, \ j = \overline{3, M-2};$$

$$b_{M-1} = B^{M-1}_{l,k}F^{M-1}_{l,k} - \alpha\frac{k_2}{h_t^2}C_M;$$

$$b_M = B^M_{l,k}F^M_{l,k} + 2\alpha\frac{k_2}{h_t^2}C_M.$$

The system (2.11) is solved by a nonmonotonic double-sweep (modified Gaussian elimination) procedure [15] with iterations on the coefficients. The iterative process is terminated when the error ε falls within predetermined limits. The solution of the system (2.11) for a fixed regularization parameter α gives the required regularized temperature profile $T^j_{l,k}$ ($j = \overline{1, \ldots, M}$). Once the temperature on the k-th line has been determined from Eq. (2.11), the transition is made to the (k − 1)-st line, etc., until the temperature on the line $y = 0$ is found. The temperature at the boundaries $x = 0$ and $x = b$ of the domain D_1 is determined from the finite-difference approximation of the boundary conditions (2.3) and (2.4) using the previously determined temperature field at interior nodes of the domain D_1.

If the error of specification of the input temperatures is known: $\delta = \left(\sum_{l=1}^{L}\sum_{j=1}^{M}\sigma^2_{l,j}\right)^{1/2}$, where $\sigma_{l,j}$ is the rms error of the function $T_c(x, t)$ at $x = x_l$, $t = t_j$, then the principle of the residual [1] can be used to determine the best approximation:

$$\left[\sum_{l=1}^{L}\sum_{j=1}^{M}(T^j_{l,K} - T_{cl})^2\right]^{1/2} - \delta = 0. \tag{2.12}$$

Here $T^j_{l,K}$ ($l = \overline{0, \ldots, L}$; $j = \overline{1, \ldots, M}$) is the temperature on the line $y = c$, which is obtained by solving the direct problem in the domain $D = D_1 \cup D_2$ with specification of the boundary conditions (1.3)-(1.5) and a known temperature T^j_{wl} ($l = \overline{0, \ldots, L}$; $j = \overline{1, \ldots, M}$) on the line $y = 0$. The values of T^j_{wl} are determined by solving the inverse problem in the domain D^1. The iterative solution of Eq. (2.12) is carried out by the method of chords. Once the temperature field in D has been determined, the heat flux density $q_w(x, t)$ is determined from the finite-difference analog of condition (1.7).

3. Results of Numerical Calculations. The algorithm described in this article has been tested numerically in the solution of a model problem. A FORTRAN program for the IBM PC/AT-386 computer was written for this purpose. The sample medium was a graphitic carbon material, mark ÉG-0, whose thermophysical characteristics $\lambda_x(T)$, $\lambda_y(T)$, and $C(T)$ were taken from [16]. Adiabatic conditions held at the boundaries of the domain $x = 0$, $x = b$, and $y = d$. Initial data on the temperature at the interior line $y = c$, which is needed to solve the IHC problem, and the temperature at the boundary $y = 0$ were obtained by solving the direct heat-conduction problem in the domain D by the decoupling method with the heat flux density at the boundary $y = 0$ specified according to the law $q_w(x, t) = Axt$ ($A = \text{const}$). The resulting functional dependence $q_w(x, t)$ and the temperature $T_w(x, t)$ determined from the solution of the direct heat-conduction problem were then adopted as an "exact" solution of the two-dimensional IHC boundary-value problem. The following values of the parameters were used in the numerical calculations: $b = 10^{-2}$ m; $c = 0.5 \cdot 10^{-2}$ m; $d = 10^{-2}$ m; $t_b = 0$; $t_f = 5$ sec; $T_b = 300$ K; $A = 10^8$; $h_x = 0.2 \cdot 10^{-2}$ m; $h_y = 0.5 \cdot 10^{-3}$ m; $h_t = 0.1$ sec; $C_M = 0$; $\varepsilon = 0.005$; $k_1 = 1$; $k_2 = 1$.

Figures 2 and 3 show the distributions of the temperature and the heat flux density in the longitudinal x-direction at the boundary $y = 0$ at $t = 0, 1, 2, 3, 4$, and 5 s (curves 1-6, respectively). The solid curves represent the exact solution of the two-dimensional IHC problem, and the dashed curves represent the solution obtained by the one-dimensional IHC algorithm [12] in different x cross sections. It is evident from the figures that the one-dimensional IHC algorithm produces large errors in the determination of $T_w(x, t)$ and $q_w(x, t)$, indicating that two-dimensional IHC algorithms must be used.

Figures 4 and 5 show the results of solving the IHC problem by means of the two-dimensional algorithm. Here curves 1 give the exact solution of the IHC problem, and curves 2 represent the solution obtained by the regularization algorithm on the basis of the unperturbed temperature $T_c(x, t)$, corresponding to the application of direct numerical methods ($\alpha = 0$). Good agreement is observed between the exact and numerical solutions. When the number of temporal and spatial points in the domain D_1 is doubled ($M = 100$, $L = 21$), the solution of the IHC problem remains essentially unchanged, but the computing time increases from 2 min to 4 min.

It is important to test the algorithm in the presence of perturbations of the initial data. This is done by superimposing on the temperature $T_c(x, t)$ perturbations having a sawtooth distribution and amplitudes equal to 1% of the maximum temperature. Curves 3 and 4 represent the solution obtained by direct numerical methods and by the regularization algorithm. It is evident from the figures that the direct numerical solution exhibits a distinctly unstable behavior until the heat flux density becomes negative. The regularization solution, on the other hand, is stable and yields good agreement with the exact solution.

To obtain a solution with smaller temporal and spatial steps in D_1 without sacrificing computer storage, it is recommended that the duration of the process and the geometrical domain D_1 be partitioned into several smaller time intervals and geometrical subdomains, in which the IHC problem is then solved in succession.

The numerical calculations demonstrate the efficiency and practicability of the proposed computational algorithm and program. The computing time for one version of the problem with perturbed initial data does not exceed 5 min on an IBM PC/AT-386 computer. One of the advantages of the proposed algorithm is its universality, which means that all possible physicochemical processes liable to occur in a heated reacting material can be taken into account on the basis of more complete mathematical models than those used here.

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